Received: *September 3, 1999*, Accepted: *October 4, 1999*

Strange quark mass determination from Cabibbo-suppressed tau decays

Antonio Pich

Departament de Física Teòrica, IFIC, Universitat de València - CSIC
Dr. Moliner 50, E-46100 Burjassot (València), Spain
E-mail: Antonio.Pich@uv.es

Joaquim Prades

Departamento de Física Teórica y del Cosmos, Universidad de Granada
Campus de Fuente Nueva, E-18002 Granada, Spain
E-mail: prades@ugr.es

Abstract: We present a general analysis of SU(3) breaking effects in the semi-inclusive tau hadronic width. The recent ALEPH measurements of the inclusive Cabibbo-suppressed decay width of the τ and several moments of its invariant mass distribution are used to determine the value of the strange quark mass. We obtain, in the $\overline{\mathrm{MS}}$ scheme, $m_s(M_\tau^2) = (119 \pm 24)\,\mathrm{MeV}$ to $O(\alpha_s^3)$, which corresponds to $m_s(1\,\mathrm{GeV}^2) = (164 \pm 33)\,\mathrm{MeV}$, $m_s(4\,\mathrm{GeV}^2) = (114 \pm 23)\,\mathrm{MeV}$.

Keywords: Quark Masses and SM Parameters, QCD.

JHEP10(1999)004

Contents

1. Introduction **1**

2. Theoretical framework **2**

3. Moments of the hadronic invariant mass distribution **5**

4. SU(3) breaking **5**

4.1 Dimension-two corrections 6

4.2 Dimension-four corrections 8

4.3 Higher-dimension corrections 10

5. Numerical analysis **12**

6. Phenomenological subtraction of the $J = L$ piece **13**

7. Comparison with the ALEPH analysis **14**

8. Summary **15**

A. Dimension-zero corrections **16**

B. Dimension-two corrections **17**

C. Dimension-four corrections **19**

D. Scale evolution of the $D = 4$ operators **22**

E. $D = 4$ expansion coefficients for δR_τ^{kl} **23**

1. Introduction

Among the free parameters of the Standard Model, the quark masses are the ones less precisely known. The lack of accurate measurements sensitive to quark mass effects and the theoretical uncertainties associated with the non-perturbative nature of QCD in the infrared region make quite difficult to perform reliable determinations of quark masses. In particular, the value of the strange quark mass has been a subject of great controversy in recent years.

In the last version of the Review of Particle Physics [1], the running strange quark mass at 2 GeV in the $\overline{\mathrm{MS}}$ scheme is quoted to be in between 60 MeV to 170 MeV. This wide range reflects both the uncertainties in the hadronic input needed in strange quark mass determinations within the context of QCD Sum Rules and the spread in m_s values obtained within lattice QCD calculations.

The high precision data on tau decays [2] collected at LEP and CESR provide a very powerful tool to analyse strange quark mass effects in a cleaner environment. The QCD analysis of the inclusive tau decay width has already made possible [3] an accurate measurement of the strong coupling constant at the τ mass scale, $\alpha_s(M_\tau^2)$, which complements and competes in accuracy with the high precision measurements of $\alpha_s(M_Z^2)$ performed at LEP. More recently, detailed experimental studies of the Cabibbo-suppressed width of the τ have started to become available [4, 5], which allows to initiate a systematic investigation of the corrections induced by the strange quark mass in the τ decay width. First theoretical studies were presented in [6, 7].

What makes a m_s determination from τ data very interesting is that the hadronic input does not depend on any extra hypothesis; it is a purely experimental issue, which accuracy can be systematically improved. The major part of the uncertainty will eventually come from the theoretical side. However, owing to its inclusive character, the total Cabibbo-suppressed tau decay width can be rigorously analyzed within QCD, using the Operator Product Expansion (OPE). Therefore, the theoretical input is in principle under control and the associated uncertainties can be quantified.

In the following we will compile and analyze in detail all what is presently known about quark mass corrections to the quark current correlation functions relevant in tau decay. In particular, we will investigate the size of these effects in the tau decay width and related observables, and the theoretical uncertainties of the corresponding predictions.

Even with the relatively large uncertainties one gets from the present data, we will show that the strange quark mass determination from tau decays has already an accuracy good enough to reduce substantially the range quoted by the Particle Data Group. The foreseen τ-charm and B-factories will further increase the precision of this measurement, allowing for much more detailed studies. Clearly, the τ-decay data will provide in the future a precise determination of the strange quark mass within QCD.

2. Theoretical framework

The theoretical analysis of the inclusive hadronic tau decay width [8, 9, 10] involves the two-point correlation functions

$$\begin{aligned}
\Pi^{\mu\nu}_{ij,V}(q) &\equiv i\int \mathrm{d}^4x\, e^{iqx}\, \langle 0|T\left\{V^\mu_{ij}(x)\, V^\nu_{ij}(0)^\dagger\right\}|0\rangle\,, \\
\Pi^{\mu\nu}_{ij,A}(q) &\equiv i\int \mathrm{d}^4x\, e^{iqx}\, \langle 0|T\left\{A^\mu_{ij}(x)\, A^\nu_{ij}(0)^\dagger\right\}|0\rangle\,,
\end{aligned} \tag{2.1}$$

associated with the vector, $V^{\mu}_{ij}(x) \equiv \overline{q}_j \gamma^{\mu} q_i$, and axial-vector, $A^{\mu}_{ij}(x) \equiv \overline{q}_j \gamma^{\mu}\gamma_5 q_i$, colour-singlet quark currents. The subscripts i, j denote the corresponding light quark flavours (up, down, and strange). These correlators have the Lorentz decompositions

$$\Pi^{\mu\nu}_{ij,V/A}(q) \;=\; \left(-g^{\mu\nu}\, q^2 + q^{\mu} q^{\nu}\right)\, \Pi^{T}_{ij,V/A}(q^2) + q^{\mu} q^{\nu}\, \Pi^{L}_{ij,V/A}(q^2)\,, \tag{2.2}$$

where the superscript in the transverse and longitudinal components denotes the corresponding angular momentum $J = 1$ (T) and $J = 0$ (L) in the hadronic rest frame.

The imaginary parts of the two-point functions $\Pi^{J}_{ij,V/A}(s)$ are proportional to the spectral functions for hadrons with the corresponding quantum numbers. The semi-hadronic decay rate of the τ lepton,

$$R_\tau \equiv \frac{\Gamma\left[\tau^- \to \text{hadrons}\; \nu_\tau(\gamma)\right]}{\Gamma\left[\tau^- \to e^-\, \overline{\nu}_e\, \nu_\tau(\gamma)\right]}\,, \tag{2.3}$$

can be expressed as an integral of the spectral functions $\operatorname{Im}\Pi^T(s)$ and $\operatorname{Im}\Pi^L(s)$ over the invariant mass s of the final-state hadrons as follows:

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{\mathrm{d}s}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \operatorname{Im}\Pi^T(s) + \operatorname{Im}\Pi^L(s)\right]. \tag{2.4}$$

The appropriate combinations of two-point correlation functions are

$$\Pi^J(s) \equiv |V_{ud}|^2 \left[\Pi^J_{ud,V}(s) + \Pi^J_{ud,A}(s)\right] + |V_{us}|^2 \left[\Pi^J_{us,V}(s) + \Pi^J_{us,A}(s)\right], \tag{2.5}$$

with $|V_{ij}|$ the corresponding Cabibbo-Kobayashi-Maskawa (CKM) quark mixing factors.

Experimentally, one can disentangle vector from axial-vector Dirac structures in the Cabibbo-allowed ($\bar{u}d$) flavour decays. Since G-parity is not a good quantum number for modes with strange particles, this separation is problematic for the Cabibbo-suppressed decays. It is then convenient to decompose the predictions for R_τ into the following three categories:

$$R_\tau \equiv R_{\tau,V} + R_{\tau,A} + R_{\tau,S}\,. \tag{2.6}$$

Where $R_{\tau,V}$ and $R_{\tau,A}$ correspond to the the two terms proportional to $|V_{ud}|^2$ in (2.5) and $R_{\tau,S}$ contains the remaining $|V_{us}|^2$ contributions.

Exploiting the analytic properties of $\Pi^J(s)$, we can express (2.4) as a contour integral in the complex s-plane running counter-clockwise around the circle $|s| = M_\tau^2$:

$$R_\tau = -\pi i \oint_{|s|=M_\tau^2} \frac{\mathrm{d}s}{s} \left(1 - \frac{s}{M_\tau^2}\right)^3 \left\{3\left(1 + \frac{s}{M_\tau^2}\right) D^{L+T}(s) + 4\, D^L(s)\right\}. \tag{2.7}$$

JHEP10(1999)004

We have used integration by parts to rewrite R_τ in terms of the logarithmic derivatives of the relevant correlators,

$$D^{L+T}(s) \equiv -s \frac{\mathrm{d}}{\mathrm{d}s} \left[\Pi^{L+T}(s)\right] ,$$
$$D^{L}(s) \equiv \frac{s}{M_\tau^2} \frac{\mathrm{d}}{\mathrm{d}s} \left[s\,\Pi^{L}(s)\right] , \tag{2.8}$$

which satisfy homogeneous renormalization group (RG) equations. In this way, one eliminates unwanted (renormalization-scheme and scale dependent) subtraction constants, which do not contribute to any physical observable.

For large enough $-s$, the contributions to $D^J(s)$ can be organized using the OPE in a series of local gauge-invariant scalar operators of increasing dimension $D = 2n$, times the appropriate inverse powers of $-s$. This expansion is expected to be well behaved along the complex contour $|s| = M_\tau^2$, except in the crossing point with the positive real axis [11]. As shown in eq. (2.7), the region near the physical cut is strongly suppressed by a zero of order three at $s = M_\tau^2$. Therefore, the uncertainties associated with the use of the OPE near the time-like axis are expected to be very small. Inserting this series in (2.7) and evaluating the contour integral, one can rewrite R_τ as an expansion in inverse powers of M_τ^2 [8],

$$R_\tau \equiv 3\left[|V_{ud}|^2 + |V_{us}|^2\right] S_{\mathrm{EW}} \left\{1 + \delta'_{\mathrm{EW}} + \delta^{(0)} + \sum_{D=2,4,\cdots} \left(\cos^2\theta_C \delta_{ud}^{(D)} + \sin^2\theta_C \delta_{us}^{(D)}\right)\right\} , \tag{2.9}$$

where $\sin^2\theta_C \equiv |V_{us}|^2/\left[|V_{ud}|^2 + |V_{us}|^2\right]$ and we have pulled out the electroweak corrections $S_{\mathrm{EW}} = 1.0194$ [12] and $\delta'_{\mathrm{EW}} \simeq 0.0010$ [13].

The dimension-zero contribution $\delta^{(0)}$ is the purely perturbative correction, neglecting quark masses, which, owing to chiral symmetry, is identical for the vector and axial-vector parts. The symbols $\delta_{ij}^{(D)} \equiv [\delta_{ij,V}^{(D)} + \delta_{ij,A}^{(D)}]/2$ stand for the average of the vector and axial-vector contributions from dimension $D \geq 2$ operators; they contain an implicit suppression factor $1/M_\tau^D$.

A general analysis of the relevant $\delta_{ij,V/A}^{(D)}$ contributions was resented in ref. [8]. A more detailed study of the perturbative piece $\delta^{(0)}$ was later performed in ref. [14], where a resummation of higher-order corrections induced by running effects along the integration contour was achieved with RG techniques. There are also studies which add renormalon-like corrections [15]. More recently, the leading quark-mass corrections of dimension two have been investigated in ref. [6]; these contributions are the dominant SU(3) breaking effect, which generates the wanted sensitivity to the strange quark mass.

In order to simplify the presentation, we will relegate a detailed compilation of the different contributions to R_τ in the appendices.

3. Moments of the hadronic invariant mass distribution

The measurement of the invariant mass distribution of the final hadrons provides additional information on the QCD dynamics. The moments [16]

$$R_\tau^{kl} \equiv \int_0^{M_\tau^2} ds \left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{dR_\tau}{ds} \qquad (3.1)$$

can be calculated theoretically in the same way as $R_\tau \equiv R_\tau^{00}$. The corresponding contour integrals can be written as

$$R_\tau^{kl} = -\pi i \oint_{|x|=1} \frac{\mathrm{d}x}{x} \left\{ 3\, \mathcal{F}_{L+T}^{kl}(x)\, D^{L+T}(M_\tau^2 x) + 4\, \mathcal{F}_L^{kl}(x)\, D^L(M_\tau^2 x) \right\}, \qquad (3.2)$$

where all kinematical factors have been absorbed into the kernels

$$\begin{aligned} \mathcal{F}_{L+T}^{kl}(x) &\equiv 2\,(1-x)^{3+k} \sum_{n=0}^{l} \frac{l!}{(l-n)!\,n!}\,(x-1)^n\, \frac{(6+k+n) + 2(3+k+n)x}{(3+k+n)(4+k+n)}\,, \\ \mathcal{F}_L^{kl}(x) &\equiv 3\,(1-x)^{3+k} \sum_{n=0}^{l} \frac{l!}{(l-n)!\,n!}\, \frac{(x-1)^n}{3+k+n}\,. \end{aligned} \qquad (3.3)$$

Table 1 shows the explicit form of these kernels for the moments which we are going to analyze in the following sections.

One can rewrite the moments R_τ^{kl} as an expansion in inverse powers of M_τ^2, analogously to (2.9). The corresponding contributions from dimension D operators will be denoted by $\delta_{ij}^{kl(D)}$.

(k,l)	$\mathcal{F}_{L+T}^{kl}(x)$	$\mathcal{F}_L^{kl}(x)$
(0,0)	$(1-x)^3\,(1+x)$	$(1-x)^3$
(1,0)	$\frac{1}{10}\,(1-x)^4\,(7+8x)$	$\frac{3}{4}\,(1-x)^4$
(2,0)	$\frac{2}{15}\,(1-x)^5\,(4+5x)$	$\frac{3}{5}\,(1-x)^5$
(1,1)	$\frac{1}{6}\,(1-x)^4\,(1+2x)^2$	$\frac{3}{20}\,(1-x)^4\,(1+4x)$
(1,2)	$\frac{1}{210}\,(1-x)^4\,(13+52x+130x^2+120x^3)$	$\frac{1}{20}\,(1-x)^4\,(1+4x+10x^2)$

Table 1: Explicit values of the relevant kinematical kernels.

4. SU(3) breaking

The separate measurement of the Cabibbo-allowed and Cabibbo-suppressed decay widths of the τ [4] allows one to pin down the SU(3) breaking effect induced by the strange quark mass, through the differences

$$\delta R_\tau^{kl} \equiv \frac{R_{\tau,V+A}^{kl}}{|V_{ud}|^2} - \frac{R_{\tau,S}^{kl}}{|V_{us}|^2} = 3\, S_{EW} \sum_{D\geq 2} \left[\delta_{ud}^{kl\,(D)} - \delta_{us}^{kl\,(D)}\right]. \qquad (4.1)$$

These observables vanish in the SU(3) limit, which helps to reduce many theoretical uncertainties. In particular they are free of possible (flavour-independent) instanton and/or renormalon contributions which could mimic dimension two corrections.

JHEP10(1999)004

4.1 Dimension-two corrections

The dimension-two corrections to the τ hadronic width are perturbative contributions proportional to m_q^2. We studied in a previous paper [6] the uncertainties associated with these corrections, in the limit $m_u = m_d = 0$. The main conclusion of that work was the relatively large uncertainty in the prediction of these corrections arising from the very bad behaviour of the $J = L$ component. We give here the general result, for arbitrary light quark masses.

In terms of the running quark masses and the QCD coupling, the $D = 2$ contributions to the correlation functions take the form

$$\begin{aligned} D^{L+T}_{ij,V/A}(s)\bigg|_{D=2} &= \frac{3}{4\pi^2 s}\Bigg\{\Big[m_i^2(-\xi^2 s) + m_j^2(-\xi^2 s)\Big]\sum_{n=0}\tilde{c}_n^{L+T}(\xi)\, a^n(-\xi^2 s) \pm \\ &\qquad \pm\, m_i(-\xi^2 s)\, m_j(-\xi^2 s)\sum_{n=1}\tilde{e}_n^{L+T}(\xi)\, a^n(-\xi^2 s) + \\ &\qquad + \Big[\sum_k m_k^2(-\xi^2 s)\Big]\sum_{n=2}\tilde{f}_n^{L+T}(\xi)\, a^n(-\xi^2 s)\Bigg\}, \end{aligned} \tag{4.2}$$

$$D^{L}_{ij,V/A}(s)\bigg|_{D=2} = -\frac{3}{8\pi^2 M_\tau^2}\Big[m_i(-\xi^2 s) \mp m_j(-\xi^2 s)\Big]^2 \sum_{n=0}\tilde{d}_n^{L}(\xi)\, a^n(-\xi^2 s)\,, \tag{4.3}$$

where $a \equiv \alpha_s/\pi$ and ξ is an arbitrary scale factor of order unity. The coefficients $\tilde{c}_n^{L+T}(\xi)$, $\tilde{e}_n^{L+T}(\xi)$, $\tilde{f}_n^{L+T}(\xi)$, and $\tilde{d}_n^{L}(\xi)$ are constrained by the RG equations satisfied by the corresponding $D^J(s)$ functions (in fact all of them follow the same RG equations). Their scale dependence is given in appendix B, together with the values of the presently known coefficients in the $\overline{\mathrm{MS}}$ scheme.

Inserting these expressions into the contour integral (3.2), the corresponding $D = 2$ contributions to R_τ^{kl}, $\delta^{kl\,(2)}_{ij,V/A}$, are given by analogous expansions, with the running coupling $a^n(-\xi^2 s)$ replaced by the functions:

$$B^{kl\,(n)}_{L+T}(a_\xi) \equiv \frac{-1}{4\pi i}\oint_{|x|=1}\frac{dx}{x^2}\,\mathcal{F}^{kl}_{L+T}(x)\left[\frac{m(-\xi^2 M_\tau^2 x)}{m(M_\tau^2)}\right]^2 a^n(-\xi^2 M_\tau^2 x)\,, \tag{4.4}$$

$$B^{kl\,(n)}_{L}(a_\xi) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\,\mathcal{F}^{kl}_{L}(x)\left[\frac{m(-\xi^2 M_\tau^2 x)}{m(M_\tau^2)}\right]^2 a^n(-\xi^2 M_\tau^2 x)\,. \tag{4.5}$$

These integrals only depend on $a_\xi \equiv \alpha_s(\xi^2 M_\tau^2)/\pi$, $\log(\xi)$, and the expansion coefficients β_i and γ_j of the QCD beta and gamma functions. They were already studied in ref. [6] for the case $(k,l) = (0,0)$.

We only need the contribution to δR_τ^{kl}, which is given by

$$\delta R_\tau^{kl}\bigg|_{D=2} = 24\, S_{EW}\,\frac{m_s^2(M_\tau^2)}{M_\tau^2}\,\left(1-\epsilon_d^2\right)\,\Delta^{(2)}_{kl}(a_\tau)\,, \tag{4.6}$$

where $a_\tau \equiv \alpha_s(M_\tau^2)/\pi$, $\epsilon_d \equiv m_d/m_s = 0.053 \pm 0.002$ [17] and[1]

$$\begin{aligned}\Delta_{kl}^{(2)}(a_\tau) &= \frac{1}{4}\left\{3 \sum_{n=0} \tilde{c}_n^{L+T}(\xi)\, B_{L+T}^{kl\,(n)}(a_\xi) + \sum_{n=0} \tilde{d}_n^{L}(\xi)\, B_{L}^{kl\,(n)}(a_\xi)\right\} \\ &\equiv \frac{1}{4}\left\{3\,\Delta_{kl}^{L+T}(a_\tau) + \Delta_{kl}^{L}(a_\tau)\right\} . \end{aligned} \tag{4.7}$$

The longitudinal series $\Delta_{kl}^{L}(a_\tau)$ is unfortunately quite problematic. The bad perturbative behaviour od $D_{ij,V/A}^{L}(s)\Big|_{D=2}$ gets reinforced by running effects along the integration contour, giving rise to a badly defined series. The convergence can be improved [6] by fully keeping the known four-loop information on the function integrals $B_J^{kl\,(n)}(a_\xi)$, i.e. using in eqs. (4.4) and (4.5) the exact solution for $m(-\xi^2 s)$ and $a(-\xi^2 s)$ obtained from the RG equations. This "contour-improved" prescription [14] allows us to resum the most important higher-order corrections, but the resulting "improved" series is still rather badly behaved. For instance,

$$\Delta_{00}^{L}(0.1) = 1.5891 + 1.1733 + 1.1214 + 1.2489 + \cdots \tag{4.8}$$

which has $O(a^2)$ and $O(a^3)$ contributions of the same size. On the contrary, the $J = L + T$ series converges very well:

$$\Delta_{00}^{L+T}(0.1) = 0.7824 + 0.2239 + 0.0831 - 0.0000601\, c_3^{L+T} + \cdots \tag{4.9}$$

Fortunately, the longitudinal contribution to $\Delta_{kl}^{(2)}(a_\tau)$ is parametrically suppressed by a factor 1/3. Thus, the combined final expansion looks still acceptable for the first few terms:

$$\Delta_{00}^{(2)}(0.1) = 0.9840 + 0.4613 + 0.3427 + \left(0.3122 - 0.000045\, c_3^{L+T}\right) + \cdots \tag{4.10}$$

Nevertheless, after the third term the series appears to be dominated by the longitudinal contribution, and the bad perturbative behaviour becomes again manifest. Using $c_3^{L+T} \sim c_2^{L+T}\left(c_2^{L+T}/c_1^{L+T}\right) \approx 323$, the fourth term becomes 0.298; i.e. a 5% reduction only. We can then take the size of the $O(a^3)$ contribution to Δ_{kl}^{L} as an educated estimate of the perturbative uncertainty.

The final numerical values of the relevant perturbative expansions are shown in table 2. We have used the value of the strong coupling constant determined by the total hadronic decay width [3]:

$$\alpha_s(M_\tau^2) \;=\; 0.35 \pm 0.02\,. \tag{4.11}$$

Two different errors are quoted in table 2. The first one gives the estimated theoretical uncertainties for the central value of $\alpha_s(M_\tau^2)$, while the second one shows the changes induced by the present uncertainty in the strong coupling.

[1]Notice that Δ_{00}^{L+T} is slightly different from the analogous quantity Δ^{L+T} defined in ref. [6] with $\tilde{d}_n^{L+T}(\xi) \equiv \tilde{c}_n^{L+T}(\xi) + \tilde{f}_n^{L+T}(\xi)$. The SU(3) singlet component $\tilde{f}_n^{L+T}(\xi)$ drops out in δR_τ^{kl}.

(k,l)	$\Delta_{kl}^{L+T}(a_\tau)$	$\Delta_{kl}^{L}(a_\tau)$	$\Delta_{kl}^{(2)}(a_\tau)$
(0,0)	$0.97 \pm 0.10 \pm 0.07$	$5.1 \pm 2.1 \pm 0.5$	$2.0 \pm 0.5 \pm 0.1$
(1,0)	$1.37 \pm 0.12 \pm 0.05$	$5.3 \pm 2.5 \pm 0.7$	$2.4 \pm 0.7 \pm 0.1$
(2,0)	$1.70 \pm 0.30 \pm 0.09$	$5.8 \pm 3.2 \pm 0.8$	$2.7 \pm 1.0 \pm 0.2$
(1,1)	$-0.37 \pm 0.11 \pm 0.02$	$-0.45 \pm 0.66 \pm 0.25$	$-0.39 \pm 0.25 \pm 0.07$
(1,2)	$0.02 \pm 0.03 \pm 0.01$	$0.23 \pm 0.17 \pm 0.08$	$0.07 \pm 0.06 \pm 0.02$

Table 2: Numerical values of the relevant $D = 2$ perturbative expansions for $\alpha_s(M_\tau^2) = 0.35 \pm 0.02$. The first error shows the estimated theoretical uncertainties taking $\alpha_s = 0.35$; the second one shows the changes induced by the present uncertainty in the strong coupling.

Since the longitudinal series (4.8) seems to reach an asymptotic behaviour at $O(a^3)$, we have taken the following criteria in our numerical estimates. The central values of $\Delta_{kl}^{(2)}(a_\tau)$ have been evaluated adding to the fully known $O(a^2)$ result one half of the $O(a^3)$ contribution. The $O(a^3)$ running effects in the L+T contribution have been also included; the remaining $O(a^3)$ contribution from the unknown constant c_3^{L+T} was estimated above to be less than 1% in $\Delta_{00}^{(2)}$. To estimate the associated theoretical uncertainties, we have taken one half of the size of the last known perturbative contribution plus the variation induced by a change of the renormalization scale in the range $\xi \in [0.75, 2]$ (added in quadrature). Finally the central values have been obtained by symmetrizing the error bars.

Notice from table 2 that the $L+T$ series is more sensitive to the value of the moment parameter k than the L series. On the other side, the two last moments with $l \neq 0$ give rise to perturbative expansions for $\Delta_{kl}^{(2)}(a_\tau)$ which are clearly unreliable; therefore, we will discard these two moments in our final m_s analysis.

4.2 Dimension-four corrections

The SU(3)-breaking piece of the $D = 4$ contribution to the correlation functions is given by

$$s^2 \left[D_{ud,V+A}^{L+T}(s) - D_{us,V+A}^{L+T}(s)\right]_{D=4} = -4\delta O_4(-\xi^2 s) \sum_{n=0} \tilde{q}_n^{L+T}(\xi) a^n(-\xi^2 s) + \\ + \frac{6}{\pi^2} m_s^4(-\xi^2 s) \left(1 - \epsilon_d^2\right) \sum_{n=0} \left\{ \left(1 + \epsilon_d^2\right) \tilde{h}_n^{L+T}(\xi) - \epsilon_u^2 \tilde{g}_n^{L+T}(\xi) \right\} a^n(-\xi^2 s)\,, \tag{4.12}$$

$$s\, M_\tau^2 \left[D_{ud,V+A}^{L}(s) - D_{us,V+A}^{L}(s)\right]_{D=4} = 2\delta O_4(-\xi^2 s) - \frac{3}{\pi^2} m_s^4(-\xi^2 s) \left(1 - \epsilon_d^2\right) \times \\ \times \sum_{n=0} \left\{ \left(1 + \epsilon_d^2\right) \left[\tilde{h}_n^L(\xi) + \tilde{j}_n^L(\xi)\right] + \epsilon_u^2 \left[2\tilde{h}_n^L(\xi) - 3\tilde{k}_n^L(\xi) + \tilde{j}_n^L(\xi)\right] \right\} a^n(-\xi^2 s)\,, \tag{4.13}$$

(k,l)	$T_{kl}(a_\tau)$	$S_{kl}(a_\tau)$	$Q_{kl}(a_\tau)$
(0,0)	$1.5 \pm 0.5 \pm 0.1$	$1.0 \pm 0.2 \pm 0.05$	$1.08 \pm 0.03 \pm 0.01$
(1,0)	$3.2 \pm 0.7 \pm 0.2$	$1.9 \pm 0.6 \pm 0.2$	$1.52 \pm 0.03 \pm 0.01$
(2,0)	$5.2 \pm 0.9 \pm 0.3$	$3.5 \pm 1.4 \pm 0.2$	$1.93 \pm 0.02 \pm 0.003$
(1,1)	$-2.0 \pm 0.3 \pm 0.2$	$-1.5 \pm 0.3 \pm 0.1$	$-0.41 \pm 0.02 \pm 0.005$
(1,2)	$0.45 \pm 0.20 \pm 0.20$	$0.3 \pm 0.1 \pm 0.1$	$-0.02 \pm 0.01 \pm 0.003$

Table 3: Numerical values of the relevant $D = 4$ perturbative expansions for $\alpha_s(M_\tau^2) = 0.35 \pm 0.02$. The first error shows the estimated theoretical uncertainties taking $\alpha_s = 0.35$; the second one shows the changes induced by the present uncertainty in the strong coupling.

where $\epsilon_u \equiv m_u/m_s = 0.029 \pm 0.003$ [17], ϵ_d was defined before, and

$$\delta O_4(\mu^2) \;\equiv\; \langle 0| m_s\, \bar{s}s - m_d\, \bar{d}d |0\rangle (\mu^2)\,. \tag{4.14}$$

The quark condensates are defined in the $\overline{\mathrm{MS}}$-scheme, at the scale $\mu^2 = -\xi^2 s$. The perturbative expansion coefficients $\tilde{q}_n^{L+T}(\xi)$, $\tilde{h}_n^{L+T}(\xi)$, $\tilde{g}_n^{L+T}(\xi)$, $\tilde{h}_n^{L}(\xi)$, $\tilde{k}_n^{L}(\xi)$, and $\tilde{j}_n^{L}(\xi)$ are given in appendix C.

Inserting these expressions into the contour formula (3.2), one gets the corresponding contributions to δR_τ^{kl}. They can be written in the form

$$\delta R_\tau^{kl}\Big|_{D=4} = 12 S_{EW} \left\{ 3\, \frac{m_s^4(M_\tau^2)}{M_\tau^4} (1-\epsilon_d^2) \left[(1+\epsilon_d^2) T_{kl}(a_\tau) - 2\epsilon_u^2 S_{kl}(a_\tau) \right] - \right.$$
$$\left. -\; 4\pi^2 \frac{\delta O_4(M_\tau^2)}{M_\tau^4} Q_{kl}(a_\tau) \right\}. \tag{4.15}$$

The normalization of the perturbative expansions $T_{kl}(a_\tau)$, $S_{kl}(a_\tau)$, and $Q_{kl}(a_\tau)$ has been chosen so that, for the lowest-order moments, these quantities are just one at leading order, i.e. $T_{00}(0) = S_{00}(0) = Q_{00}(0) = 1$. Their explicit expressions are given in appendix E, for the (k,l) values which are going to be relevant in our analysis. Table 3 here and tables 9, and 10 in appendix E show their corresponding numerical values.

In principle, the SU(3)-breaking condensate $\delta O_4(M_\tau^2)$ could be extracted from the τ decay data, together with m_s, through a combined fit of different δR_τ^{kl} moments. However, this is not possible with the actual experimental accuracy. In the future this could be the best determination of the SU(3)-breaking condensate $\delta O_4(M_\tau^2)$.

We can estimate the value of $\delta O_4(M_\tau^2)$ using the constraints provided by the chiral symmetry of QCD. To lowest order in Chiral Perturbation Theory [18], $\delta O_4(\mu^2)$ is scale independent and is fully predicted in terms of the pion decay constant and the pion and kaon masses:

$$\delta O_4(M_\tau^2)\Big|_{O(p^2)} = (m_s - m_d)\, \langle 0|\bar{q}q|0\rangle \simeq -f_\pi^2 \left(m_K^2 - m_\pi^2 \right) \simeq -1.9\times 10^{-3}\,\mathrm{GeV}^4\,. \tag{4.16}$$

Here, $\langle 0|\overline{q}q|0\rangle$ denotes the quark condensate in the chiral limit, which we take to be approximately given by $2\hat{m}\,\langle 0|\overline{q}q|0\rangle \simeq \hat{m}\,\langle 0|\overline{d}d+\overline{u}u|0\rangle \simeq -f_\pi^2 m_\pi^2$ [18, 19], where $\hat{m} \equiv (m_u+m_d)/2$.

We can improve this estimate, taking into account the leading $O(p^4)$ corrections through the ratio of quark vacuum condensates,[2]

$$v_s \equiv \frac{\langle 0|\overline{s}s|0\rangle}{\langle 0|\overline{d}d|0\rangle} = 0.8 \pm 0.2\,. \tag{4.17}$$

This ratio has been phenomenologically estimated to be around $0.6 \sim 0.9$ for scales between 1 and 2 GeV where the scale dependence is very soft [20, 21, 22]. To be conservative, we have enlarged slightly its allowed range to include the SU(3) symmetric value $v_s = 1$. Taking this correction into account, we get our final estimate

$$\begin{aligned}\delta O_4(M_\tau^2)\Big|_{O(p^4)} &= (v_s\, m_s - m_d)\;\langle 0|\bar{d}d|0\rangle \;\simeq\; -\frac{m_s}{2\hat{m}}\,(v_s-\epsilon_d)\; f_\pi^2\, m_\pi^2 \\ &\simeq -(1.5\pm 0.4)\times 10^{-3}\,\mathrm{GeV}^4\,,\end{aligned} \tag{4.18}$$

where we have used the known values of quark mass ratios [17] $m_s/\hat{m} = 24.4 \pm 1.5$ and ϵ_d given before. At this order, $\delta O_4(\mu^2)$ is still scale independent in QCD.

For typical hadronic scales the quark condensate gives a sizeable contribution to δR_τ^{kl}, proportional to $-4\pi^2 \delta O_4(M_\tau^2)/M_\tau^4 \approx (5.9\pm 1.6)\times 10^{-3}$, which is much larger than the remaining $O(m^4)$ corrections. Those m^4 contributions are of the same order than the scale dependence of $\delta O_4(\mu^2)$, which is smaller than the accuracy of our estimate (4.18). To be consistent, we will therefore neglect all m^4 corrections together with the scale dependence of $\delta O_4(\mu^2)$.

4.3 Higher-dimension corrections

The leading order coefficients of dimension six and eight corrections for two point functions have been studied in the $\overline{\mathrm{MS}}$ scheme [23]–[30]. However, in view of the theoretical uncertainties in the dimension two and four corrections and the unknown values of the dimension six and eight condensates, we shall not include the $D \geq 6$ contributions and regard them as an additional theoretical uncertainty.

To get an order of magnitude estimate of the possible size of those effects, let us neglect their logarithmic dependences and parameterize the leading $D = 6$ contributions to the two-point correlators as

$$\begin{aligned} s^3\,\left[D^{L+T}_{ud,V+A}(s) - D^{L+T}_{us,V+A}(s)\right]_{D=6} &= -3\,\delta O_6^{L+T}\,, \\ s^2\,M_\tau^2\,\left[D^{L}_{ud,V+A}(s) - D^{L}_{us,V+A}(s)\right]_{D=6} &= 2\,\delta O_6^{L}\,. \end{aligned} \tag{4.19}$$

[2]Strictly speaking this ratio is UV scale dependent in QCD. As shown in appendix D, this dependence is canceled by m^4 terms and is then of order p^8 in the momentum expansion. For typical hadronic scales, these $O(m^4)$ corrections are very small and will be neglected in the following.

The corresponding contribution to the different δR_τ^{kl} moments is then:

$$
\begin{aligned}
\delta R_\tau^{00}\Big|_{D=6} &\approx -\frac{12\pi^2}{M_\tau^6}\,\left[3\,\delta O_6^{L+T} - 4\,\delta O_6^{L}\right],\\
\delta R_\tau^{10}\Big|_{D=6} &\approx -\frac{12\pi^2}{M_\tau^6}\,\left[3\,\delta O_6^{L+T} - 6\,\delta O_6^{L}\right],\\
\delta R_\tau^{20}\Big|_{D=6} &\approx -\frac{12\pi^2}{M_\tau^6}\,\left[2\,\delta O_6^{L+T} - 8\,\delta O_6^{L}\right],\\
\delta R_\tau^{11}\Big|_{D=6} &\approx -\frac{12\pi^2}{M_\tau^6}\,\left[\delta O_6^{L+T} + 2\,\delta O_6^{L}\right],\\
\delta R_\tau^{12}\Big|_{D=6} &\approx \frac{12\pi^2}{M_\tau^6}\,\delta O_6^{L+T}\,.
\end{aligned}
\tag{4.20}
$$

One could expect that the leading $D = 6$ contributions come from four-quark operators, because they are not suppressed by light quark masses (the G^3 operator is flavour symmetric and therefore does not contribute to δR_τ^{kl}). Obviously, only the $L + T$ piece gets such a contribution, which can be obtained from ref. [8] in the vacuum saturation approximation,

$$
\begin{aligned}
\delta R_\tau^{00}\Big|_{D=6} &\approx -S_{EW}\,\frac{36\pi^2}{M_\tau^6}\,\delta O_6^{L+T} \approx S_{EW}\,\frac{256\pi^4}{9}a_\tau\,\frac{\langle 0|\bar{s}s|0\rangle^2(M_\tau^2) - \langle 0|\bar{d}d|0\rangle^2(M_\tau^2)}{M_\tau^6}\\
&\approx 3\,S_{EW}\,\delta_{ud}^{00\,(D=6)}\,\frac{1-v_s^2}{2} \approx (0.6 \pm 2.3)\times 10^{-3}\,.
\end{aligned}
\tag{4.21}
$$

In that case, we also have

$$
\delta R_\tau^{00}\Big|_{D=6} \approx \delta R_\tau^{10}\Big|_{D=6} \approx \frac{3}{2}\delta R_\tau^{20}\Big|_{D=6} \approx 3\delta R_\tau^{11}\Big|_{D=6} \approx -3\delta R_\tau^{12}\Big|_{D=6}\,.
\tag{4.22}
$$

To get the final number, we have used the measured value of the Cabibbo-allowed correction [31, 32] $\delta_{ud}^{00\,(D=6)} = 0.001 \pm 0.004$.

The size of dimension six corrections proportional to four-quark operators is smaller than the uncertainty from the dimension four corrections for the three moments that we are going to use $[(k,l) = (0,0),(1,0),(2,0)]$. We thus conclude that dimension six and higher corrections are negligible given the actual experimental accuracy (see table 4) and do not add any further uncertainty to the m_s determination at present if one uses the three moments above. They will become eventually important for the determination of the SU(3)-breaking condensate $\delta O_4(M_\tau^2)$.

Notice that higher-order corrections could be important and even dominant in some cases. For instance, in the moment $(k,l) = (1,2)$ there is a strong suppression of the contributions with dimensions two and four (see tables 2 and 3), which makes necessary to consider the $D = 6$ terms.

JHEP10(1999)004

5. Numerical analysis

Discarding $O(m^4)$ corrections, which are much smaller than the present experimental uncertainties, and up to dimension six corrections, which are around eight times smaller than the uncertainty in the dimension four contribution for the three moments $[(k,l) = (0,0), (1,0), (2,0)]$ considered here,

$$m_s^2(M_\tau^2) = \frac{M_\tau^2}{2(1-\epsilon_d^2)} \frac{1}{\Delta_{kl}^{(2)}(a_\tau)} \left[\frac{\delta R_\tau^{kl}}{12 S_{EW}} + 4\pi^2 \frac{\delta O_4(M_\tau^2)}{M_\tau^4} Q_{kl}(a_\tau) \right]. \tag{5.1}$$

The ALEPH collaboration has measured [4] the weighted differences δR_τ^{kl} for five different values of (k,l). The experimental results are shown in table 4, together with the corresponding $m_s(M_\tau^2)$ values. Since the QCD counterparts to the moments (k,l) = (1,1) and (1,2) have theoretical uncertainties larger than 100 %, we only use the moments (k,l) = (0,0), (1,0), and (2,0).

(k,l)	δR_τ^{kl}	$m_s(M_\tau^2)$ (MeV)
(0,0)	0.394 ± 0.137	$143 \pm 31 \pm 18$
(1,0)	0.383 ± 0.078	$121 \pm 17 \pm 18$
(2,0)	0.373 ± 0.054	$106 \pm 12 \pm 21$
(1,1)	0.010 ± 0.029	-
(1,2)	0.006 ± 0.015	-

Table 4: Measured [4] values of δR_τ^{kl} and the strange quark mass at M_τ. The first error is experimental and the second is from the uncertainty in the QCD counterpart.

The experimental errors quoted in table 4 do not include the present uncertainty in $|V_{us}|$. To estimate the corresponding error in m_s, we take the following numbers published by ALEPH [4, 31]: $R_{\tau,V+A}^{00} = 3.486 \pm 0.015$, $R_{\tau,S}^{00} = 0.1610 \pm 0.0066$, $|V_{ud}| = 0.9751 \pm 0.0004$ and $|V_{us}| = 0.2218 \pm 0.0016$. This gives[3] $\delta R_\tau^{00} = 0.394 \pm 0.135 \pm 0.047$, where the second error comes from the uncertainty in $|V_{us}|$ and translates into an additional uncertainty of 10 MeV in the strange quark mass. Since the ALEPH collaboration does not quote the values of $R_{\tau,V+A}^{kl}$ and $R_{\tau,S}^{kl}$ for the other moments, we will put the same $|V_{us}|$ uncertainty to the other two moments.

Taking the information from the three moments into account, we get our final result for $m_s(M_\tau^2)$:

$$m_s(M_\tau^2) = (119 \pm 12 \pm 18 \pm 10)\,\text{MeV} = (119 \pm 24)\,\text{MeV} \tag{5.2}$$

The first error is experimental, the second reflects the QCD uncertainty and the third one is from the present uncertainty in $|V_{us}|$. Since the three moments are highly correlated, we have taken the smaller individual errors as errors of the final average.

[3] Using the PDG98 [1] values (with the constraint $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$) $|V_{ud}| = 0.97525 \pm 0.00046$ and $|V_{us}| = 0.2211 \pm 0.0020$, we get $\delta R_\tau^{00} = 0.372 \pm 0.136 \pm 0.060$, which lowers 5 MeV the central value of $m_s(M_\tau^2)$ from δR_τ^{00}.

At the usual scales $\mu = 1\,\text{GeV}$ and $\mu = 2\,\text{GeV}$ (used as reference values by the QCD Sum Rules and lattice communities, respectively), our determination (5.2) corresponds to

$$m_s(1\,\text{GeV}^2) = (164 \pm 17 \pm 25 \pm 14)\,\text{MeV} \; = (164 \pm 33)\,\text{MeV} \tag{5.3}$$

and

$$m_s(4\,\text{GeV}^2) = (114 \pm 12 \pm 17 \pm 10)\,\text{MeV} \; = (114 \pm 23)\,\text{MeV}\,. \tag{5.4}$$

6. Phenomenological subtraction of the $J = L$ piece

In order to avoid the large theoretical uncertainties associated with the bad perturbative behaviour of $\Delta_{kl}^{L}(a_\tau)$, it would be nice to have "experimental values" for the $J = L+T$ contributions to δR_τ^{kl}.

Using the positivity of the spectral functions, the known pion and kaon poles provide the lower bounds

$$\begin{aligned} \text{Im}\,\Pi_{ud}^{L}(s) &\geq 2\pi f_\pi^2 \delta(s - m_\pi^2) \\ \text{Im}\,\Pi_{us}^{L}(s) &\geq 2\pi f_K^2 \delta(s - m_K^2)\,, \end{aligned} \tag{6.1}$$

(k,l)	$\delta R_{\tau,L+T}^{kl}$	$m_s(M_\tau^2)$ (MeV)
(0,0)	< 0.411	< 287
(1,0)	< 0.351	< 246
(2,0)	< 0.326	< 202
(1,1)	< 0.030	–
(1,2)	< 0.020	–

Table 5: Bounds on $\delta R_{\tau,L+T}^{kl}$ and the strange quark mass at M_τ.

which translate into upper limits on the corresponding $J = L$ contributions to R_τ^{kl},

$$\begin{aligned} R_{\tau,L}^{kl} &= -24\pi \int_0^{M_\tau^2} \frac{\mathrm{d}s}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^{2+k} \left(\frac{s}{M_\tau^2}\right)^{1+l} \text{Im}\,\Pi^{L}(s) \\ &= -4\pi i \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, \mathcal{F}_L^{kl}(x)\, D^{L}(M_\tau^2 x)\,. \end{aligned} \tag{6.2}$$

After subtracting the Goldstone boson pole, Im $\Pi_{ij}^{L}(s)$ is proportional to light quark masses squared. Since $m_s \gg m_{u,d}$, we can then safely conclude

$$\delta R_{\tau,L}^{kl} > 48\pi^2 \left[\frac{f_K^2}{M_\tau^2}\left(1 - \frac{m_K^2}{M_\tau^2}\right)^{2+k}\left(\frac{m_K^2}{M_\tau^2}\right)^{1+l} - \frac{f_\pi^2}{M_\tau^2}\left(1 - \frac{m_\pi^2}{M_\tau^2}\right)^{2+k}\left(\frac{m_\pi^2}{M_\tau^2}\right)^{1+l}\right]. \tag{6.3}$$

Subtracting this contribution from the measured values of δR_τ^{kl}, one gets upper bounds on $\delta R_{\tau,L+T}^{kl}$. Therefore, using the relation

$$m_s^2(M_\tau^2) = \frac{2M_\tau^2}{3(1-\epsilon_d^2)} \frac{1}{\Delta_{kl}^{L+T}(a_\tau)} \left[\frac{\delta R_{\tau,L+T}^{kl}}{12 S_{EW}} + 4\pi^2 \frac{\delta O_4(M_\tau^2)}{M_\tau^4} Q_{kl}^{L+T}(a_\tau)\right], \tag{6.4}$$

we can get non-trivial upper bounds on $m_s(M_\tau^2)$. The resulting numerical values are given in table 5.

To improve on these bounds it would be necessary to have a better understanding of the $J = L$ spectral functions.

7. Comparison with the ALEPH analysis

The ALEPH collaboration has also performed a phenomenological analysis of the weighted differences δR_τ^{kl} in table 4. However, the resulting values of the strange quark mass quoted by ALEPH [4] are larger:

$$m_s(M_\tau^2) \,=\, \begin{cases} (149\,{}^{+24_{\rm exp}}_{-30_{\rm exp}}\,{}^{+21_{\rm th}}_{-25_{\rm th}} \pm 6_{\rm fit})\ \mathrm{MeV} & \text{(inclusive)}, \\ (176\,{}^{+37_{\rm exp}}_{-48_{\rm exp}}\,{}^{+24_{\rm th}}_{-28_{\rm th}} \pm 8_{\rm fit} \pm 11_{J=0})\ \mathrm{MeV} & (L+T \text{ only}). \end{cases} \tag{7.1}$$

To derive these numbers, ALEPH has used our published results in references [6], [8], and [14]. Since we have analysed here the same ALEPH data with improved theoretical input, it is worthwhile to understand qualitatively the main origin of the numerical difference.

ALEPH makes a global fit to the five measured moments. Moreover, they also fit two additional parameters $\delta_S^{(6)}$ and $\delta_S^{(8)}$, trying to extract higher-order non-perturbative corrections from the data. As we have pointed out before, the last two moments have theoretical errors (in the leading $D = 2$ contribution) larger than 100% and therefore are unreliable. Unfortunately, the sensitivity to the small $D \geq 6$ corrections comes precisely from these two moments (see our comments at the end of section 4.3), which makes the fitted $\delta_S^{(6,8)}$ values very questionable.

Owing to the asymptotic behaviour of $\Delta_{kl}^L(a_\tau)$, ALEPH truncates the contribution of this perturbative series at $O(a)$ (in the "inclusive" analysis), neglecting the known (and positive) $O(a^2)$ and $O(a^3)$ longitudinal contributions.[4] Thus, they use a smaller value of $\Delta_{kl}^{(2)}(a_\tau)$ and, therefore, get a larger result for the strange quark mass,[5] because the sensitivity to this parameter is through the product $m_s^2(M_\tau^2)\,\Delta_{kl}^{(2)}(a_\tau)$. Since they put rather conservative errors, their result [the value quoted as "inclusive" in eq. (7.1)] is consistent with ours.

In order to avoid the large perturbative corrections in the longitudinal piece, the ALEPH collaboration has made a second analysis, subtracting the $J = L$ contribution in a way completely analogous to the one presented in section 6. However, besides subtracting the pion and kaon poles, ALEPH makes a tiny ad-hoc correction to account for the remaining unknown $J = L$ contribution, and quotes the resulting number as a $m_s(M_\tau^2)$ determination [the value quoted as "$L + T$ only" in eq. (7.1)]. Since they add a generous uncertainty, their number does not disagree with ours. It is clear, however, from our discussion in section 6, that this is actually an upper bound on $m_s(M_\tau^2)$ and not a determination of this parameter.

[4]The relevant quantity for this analysis is $\Delta_{kl}^{(2)}(a_\tau)$ which does not reach its miminum term before $O(a^3)$ [6] [see Eq. (4.10)].

[5]In fact, larger values of m_s were also obtained in the first determinations of this parameter from QCD sum rules, performed at $O(\alpha_s)$. Once the large higher-order perturbative corrections to the corresponding correlators were known, the resulting m_s values shifted down by a sizeable amount.

JHEP10(1999)004

8. Summary

We have analysed the SU(3) breaking effects in the semi-inclusive τ hadronic width in complete generality. This has been used to obtain the strange quark mass from the recent ALEPH measurement of the inclusive Cabibbo-suppressed decay width and several moments of its invariant mass distribution. We get

$$m_s(M_\tau^2) = (119 \pm 12 \pm 18 \pm 10)\,\mathrm{MeV} = (119 \pm 24)\,\mathrm{MeV} \tag{8.1}$$

to $O(\alpha_s^3)$ within the $\overline{\mathrm{MS}}$ scheme.[6] The first error comes from the experimental uncertainty, the second one from the uncertainty in the QCD counterpart and the third from the uncertainty in $|V_{us}|$. This determination of the strange quark mass relies on the assumption of global duality between QCD and the hadronic spectrum, which has been shown to work extremely well for the inclusive tau decay width [31, 32].

At the customary scales where quark masses are quoted, this result translates into

$$m_s(1\,\mathrm{GeV}^2) = (164 \pm 17 \pm 25 \pm 14)\,\mathrm{MeV} = (164 \pm 33)\,\mathrm{MeV} \tag{8.2}$$

and

$$m_s(4\,\mathrm{GeV}^2) = (114 \pm 12 \pm 17 \pm 10)\,\mathrm{MeV} = (114 \pm 23)\,\mathrm{MeV}\,. \tag{8.3}$$

This agrees within errors with the findings in ref. [6], where only δR_τ^{00} was used.

Subtracting the known kaon and pion poles, we have also obtained an upper bound on the strange quark mass,

$$m_s(M_\tau^2) \;<\; 202\ \mathrm{MeV}\,, \tag{8.4}$$

which corresponds to $m_s(1\,\mathrm{GeV}^2) < 277\,\mathrm{MeV}$ and $m_s(4\,\mathrm{GeV}^2) < 194\,\mathrm{MeV}$. This bound is completely free from the problems associated with the bad perturbative behaviour of the $J = L$ contribution. Our result is compatible with the lower bounds presented in refs. [33, 34, 35].

There is a great deal of activity calculating the strange quark mass by the lattice community [36]. The results are still very confusing and the spread in values obtained using different approximations to QCD is quite large. For a critical view of the situation see [37].

The latest QCD Sum Rules determinations [38]–[43] have been obtaining results which are very compatible with our number. The systematic error in those determinations is however still unclear.

The sum of the up and down quark masses has been determined with Finite Energy Sum Rules in ref. [19] with the result $(m_u + m_d)(4\,\mathrm{Ge}V^2) = (9.8 \pm 1.9)\,\mathrm{MeV}$.

[6] There is a remaining $O(a^3)$ contribution from the unknown constant c_3^{L+T}, which was estimated in section (4.1) to modify m_s by less than 0.5 %.

Using the ratio of light quark masses $2m_s/(m_u + m_d) = 24.4 \pm 1.5$, obtained within $O(p^4)$ Chiral Perturbation Theory and large N_c [17], this result is also in nice agreement with our determination

We have not made any attempt to reduce the theoretical error, which we defer to a future publication. As stated in ref. [6], once the invariant mass distribution of the final τ-decay hadrons is known, it should be possible to find weighted distributions with smaller theoretical errors for the dimension-two QCD counterpart. If the SU(3)-breaking data from tau decays is improved at future facilities, it could be the source of precise determinations of both the strange quark mass and the SU(3)-breaking condensate $\delta O_4(M_\tau^2)$.

Acknowledgments

We have benefit from useful discussions with Shaomin Chen, Michel Davier and Andreas Höcker. J.P. would like to thank the hospitality of the CERN Theory Division and the Departament de Física Teòrica at the Universitat de València (Spain) where part of his work was done. This work has been supported in part by the European Union TMR Network EURODAPHNE (Contract No. ERBFMX-CT98-0169), by CICYT, Spain, under grants No. AEN-96/1672 and PB97-1261, and by Junta de Andalucía Grant No. FQM-101.

A. Dimension-zero corrections

Though they have been extensively studied in refs. [8] and [14], for the sake of completeness, we give here also the dimension-zero corrections to R_τ^{kl}. They are flavour independent and are identical for the vector and axial-vector correlators. Moreover, only the transverse piece gets $D = 0$ contributions:

$$D(s) \equiv D^{L+T}_{ij,V/A}(s)\Big|_{D=0} = \frac{1}{4\pi^2} \sum_{n=0} \tilde{K}_n(\xi)\, a^n(-\xi^2 s)\,. \tag{A.1}$$

The coefficients $\tilde{K}_n(\xi)$ are constrained by the homogeneous RG equations satisfied by the Adler function $D(s)$:

$$\xi \frac{\mathrm{d}}{\mathrm{d}\xi} \tilde{K}_n(\xi) = \sum_{k=1}^{n} (k-n)\, \beta_k\, \tilde{K}_{n-k}(\xi)\,, \tag{A.2}$$

for $n \geq 1$, and

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \tilde{K}_0(\xi) = 0\,. \tag{A.3}$$

Thus,

$$\begin{aligned}
\tilde{K}_0(\xi) &= K_0\,, \\
\tilde{K}_1(\xi) &= K_1\,, \\
\tilde{K}_2(\xi) &= K_2 - \beta_1\, K_1 \ln\xi\,, \\
\tilde{K}_3(\xi) &= K_3 - [\beta_2\, K_1 + 2\beta_1 K_2] \ln\xi + \beta_1^2 K_1 \ln^2\xi\,, \\
\tilde{K}_4(\xi) &= K_4 - \cdots
\end{aligned} \tag{A.4}$$

The factors β_k are the expansion coefficients of the QCD beta function, defined as

$$\beta(a) \equiv \frac{\mu}{a}\frac{\mathrm{d}a}{\mathrm{d}\mu} = \sum_{k=1} \beta_k\, a^k\,, \tag{A.5}$$

which are known to four loops [44, 45]. For three flavours and in the $\overline{\mathrm{MS}}$ scheme,

$$\beta_1 = -\frac{9}{2}\,, \qquad \beta_2 = -8\,, \qquad \beta_3 = -\frac{3863}{192}\,, \qquad \beta_4 = -\frac{140599}{2304} - \frac{445}{16}\,\zeta_3\,. \tag{A.6}$$

The perturbative expansion of the Adler function is fully known up to order α_s^3. At $\xi = 1$, its coefficients have the values [46, 47]:

$$K_0 = K_1 = 1\,, \qquad K_2 = \frac{299}{24} - 9\,\zeta_3\,, \qquad K_3 = \frac{58057}{288} - \frac{779}{4}\zeta_3 - \frac{75}{2}\zeta_5\,. \tag{A.7}$$

The perturbative component of R_τ is given by

$$\delta^{kl(0)} = \sum_{n=1} \tilde{K}_n(\xi)\, A_{kl}^{(n)}(a_\xi)\,, \tag{A.8}$$

where the functions [14]

$$A_{kl}^{(n)}(a_\xi) = \frac{1}{2\pi i}\oint_{|x|=1} \frac{\mathrm{d}x}{x}\, \mathcal{F}_{L+T}^{kl}(x)\, a^n(-\xi^2 M_\tau^2 x)\,, \tag{A.9}$$

are contour integrals in the complex plane which only depend on $a_\xi \equiv \alpha_s(\xi^2 M_\tau^2)/\pi$, $\ln(\xi)$, and the expansion coefficients β_i of the QCD beta function. A detailed analysis of this contribution to R_τ and its associated uncertainty can be found in ref. [14].

B. Dimension-two corrections

The dimension-two corrections to the correlators $D_{ij,V/A}^J(s)$ are given in eqs. (4.2) and (4.3). The corresponding expansion coefficients obey the RG equation

$$\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{C}_n(\xi) = \sum_{k=1}^{n} [2\gamma_k - (n-k)\,\beta_k]\; \tilde{C}_{n-k}(\xi)\,, \tag{B.1}$$

for $n \geq 1$ and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{C}_0(\xi) = 0\,, \tag{B.2}$$

where the generic notation $\tilde{C}(\xi)$ stands for $\tilde{c}_n^{L+T}(\xi)$, $\tilde{e}_n^{L+T}(\xi)$, $\tilde{f}_n^{L+T}(\xi)$, and $\tilde{d}_n^{L}(\xi)$. Therefore,

$$\begin{aligned}
\tilde{C}_0(\xi) &= C_0\,,\\
\tilde{C}_1(\xi) &= C_1 + 2\gamma_1 C_0 \ln\xi\,,\\
\tilde{C}_2(\xi) &= C_2 + [2\gamma_2 C_0 + (2\gamma_1-\beta_1)C_1]\ln\xi + \gamma_1(2\gamma_1-\beta_1)C_0\ln^2\xi\,,\\
\tilde{C}_3(\xi) &= C_3 + [2\gamma_3 C_0 + (2\gamma_2-\beta_2)C_1 + 2(\gamma_1-\beta_1)C_2]\ln\xi +\\
&\quad + [(-\gamma_1\beta_2 + 2\gamma_2(2\gamma_1-\beta_1))C_0 + (\gamma_1-\beta_1)(2\gamma_1-\beta_1)C_1]\ln^2\xi +\\
&\quad + \frac{2}{3}\gamma_1(\gamma_1-\beta_1)(2\gamma_1-\beta_1)C_0\ln^3\xi\,,\\
\tilde{C}_4(\xi) &= C_4 + \cdots\,.
\end{aligned} \tag{B.3}$$

The factors γ_k are the expansion coefficients of the QCD gamma function, defined as

$$\gamma(a) \equiv -\frac{\mu}{m}\frac{\mathrm{d}m}{\mathrm{d}\mu} = \sum_{k=1}\gamma_k\, a^k\,, \tag{B.4}$$

which are known to four loops [45, 48, 49]. For three flavours and in the $\overline{\mathrm{MS}}$ scheme,

$$\begin{aligned}
\gamma_1 = 2\,, \qquad \gamma_2 = \frac{91}{12}\,, \qquad \gamma_3 = \frac{8885}{288} - 5\zeta_3\,,\\
\gamma_4 = \frac{2977517}{20736} - \frac{9295}{216}\zeta_3 + \frac{135}{8}\zeta_4 - \frac{125}{6}\zeta_5\,.
\end{aligned} \tag{B.5}$$

The $J = L+T$ coefficients are known to $O(\alpha_s^2)$ [50]–[56] while the $J = L$ coefficients are known to $O(\alpha_s^3)$ [50, 51, 52, 56]. Their values at $\xi = 1$ are

$$\begin{aligned}
c_0^{L+T} &= 1\,, & c_1^{L+T} &= \frac{13}{3}\,, & c_2^{L+T} &= \frac{23077}{432} + \frac{179}{54}\zeta_3 - \frac{520}{27}\zeta_5\,,\\
e_0^{L+T} &= 0\,, & e_1^{L+T} &= \frac{2}{3}\,, & e_2^{L+T} &= \frac{769}{54} - \frac{55}{27}\zeta_3 - \frac{5}{27}\zeta_5\,,\\
f_0^{L+T} &= 0\,, & f_1^{L+T} &= 0\,, & f_2^{L+T} &= -\frac{32}{9} + \frac{8}{3}\zeta_3\,,
\end{aligned} \tag{B.6}$$

and

$$\begin{aligned}
d_0^L = 1\,, \qquad d_1^L = \frac{17}{3}\,, \qquad d_2^L = \frac{9631}{144} - \frac{35}{2}\zeta_3\,,\\
d_3^L = \frac{4748953}{5184} - \frac{91519}{216}\zeta_3 - \frac{5}{2}\zeta_4 + \frac{715}{12}\zeta_5\,.
\end{aligned} \tag{B.7}$$

In the limit $m_u = m_d = 0$, taken in ref. [6], the expansion of the $J = L+T$ correlator is governed by the combinations $\tilde{d}_n^{L+T}(\xi) \equiv \tilde{c}_n^{L+T}(\xi) + \tilde{f}_n^{L+T}(\xi)$.

C. Dimension-four corrections

The dimension-four corrections to the correlators $D^J_{ij,V/A}(s)$ can be written in the form:

$$D^{L+T}_{ij,V/A}(s)\Big|_{D=4} = \frac{1}{s^2}\sum_{n=0}\tilde{\Omega}^{L+T}_n(\xi,s)\,a^n(-\xi^2 s)\,, \tag{C.1}$$

$$\begin{aligned} D^{L}_{ij,V/A}(s)\Big|_{D=4} = \frac{1}{M_\tau^2 s}\Big\{&-\langle(m_i\mp m_j)\,(\bar{q}_i q_i\mp\bar{q}_j q_j)\rangle_\xi + \\ &+\frac{3}{2\pi^2}\,(\overline{m}_i\mp\overline{m}_j)^2\sum_{n=0}\tilde{\Omega}^L_n(\xi,s)a^n(-\xi^2 s)\Big\}\,, \end{aligned} \tag{C.2}$$

where

$$\begin{aligned} \tilde{\Omega}^{L+T}_n(\xi,s) = {}&\frac{1}{6}\langle G^2\rangle_\xi\tilde{p}^{L+T}_n(\xi) + \Big\langle\sum_k m_k\,\bar{q}_k q_k\Big\rangle_\xi\,\tilde{r}^{L+T}_n(\xi) + \\ &+2\langle m_i\bar{q}_i q_i + m_j\bar{q}_j q_j\rangle_\xi\tilde{q}^{L+T}_n(\xi)\pm\frac{8}{3}\langle m_j\bar{q}_i q_i + m_i\bar{q}_j q_j\rangle_\xi\tilde{t}^{L+T}_n(\xi) - \\ &-\frac{3}{\pi^2}\Big\{\left(\overline{m}_i^4+\overline{m}_j^4\right)\tilde{h}^{L+T}_n(\xi)\pm\frac{5}{3}\,\overline{m}_i\,\overline{m}_j\left(\overline{m}_i^2+\overline{m}_j^2\right)\tilde{k}^{L+T}_n(\xi) - \\ &-\overline{m}_i^2\,\overline{m}_j^2\tilde{g}^{L+T}_n(\xi)+\sum_k\overline{m}_k^4\tilde{j}^{L+T}_n(\xi)+2\sum_{k\neq l}\overline{m}_k^2\overline{m}_l^2\tilde{u}^{L+T}_n(\xi)\Big\}\,, \end{aligned} \tag{C.3}$$

$$\tilde{\Omega}^L_n(\xi,s) = \left(\overline{m}_i^2+\overline{m}_j^2\right)\,\tilde{h}^L_n(\xi)\,\pm\,\frac{3}{2}\overline{m}_i\,\overline{m}_j\,\tilde{k}^L_n(\xi)\,+\sum_k\overline{m}_k^2\,\tilde{j}^L_n(\xi)\,. \tag{C.4}$$

In these expressions, the running masses and vacuum condensates are defined in the $\overline{\mathrm{MS}}$ scheme at the scale $\mu^2=-\xi^2 s$, i.e.

$$\overline{m}_i\equiv m_i(-\xi^2 s)\,,\qquad \langle G^2\rangle_\xi\equiv\langle 0|G^2|0\rangle(-\xi^2 s)\,,\qquad \langle m_i\bar{q}_j q_j\rangle_\xi\equiv\langle 0|m_i\bar{q}_j q_j|0\rangle(-\xi^2 s)\,. \tag{C.5}$$

In ref. [8] the $D=4$ contributions were given in terms of scale-invariant condensates. This simplifies the R_τ contour integration, but introduces inverse powers of α_s in some m^4 terms, generating larger quark-mass corrections which cancel numerically with the condensate contributions [57]. With the minimally subtracted operators (C.5), used here, one gets slightly more stable numerical results for the dimension-four mass corrections.

The quark condensate contribution to the longitudinal correlator (C.2) is fixed to all orders in perturbation theory by a Ward identity.

The perturbative expansion coefficients in eqs. (C.1) and (C.2) are known to $O(a^2)$ [28, 30, 58, 59, 60, 61, 62, 63, 64, 65] for the condensate contributions,

$$
\begin{aligned}
&\tilde{p}_0^{L+T}(\xi) = 0\,, && \tilde{p}_1^{L+T}(\xi) = 1\,, && \tilde{p}_2^{L+T}(\xi) = \frac{7}{6}\,,\\
&\tilde{r}_0^{L+T}(\xi) = 0\,, && \tilde{r}_1^{L+T}(\xi) = 0\,, && \tilde{r}_2^{L+T}(\xi) = -\frac{5}{3} + \frac{8}{3}\zeta_3 + \frac{4}{3}\ln\xi\,,\\
&\tilde{q}_0^{L+T}(\xi) = 1\,, && \tilde{q}_1^{L+T}(\xi) = -1\,, && \tilde{q}_2^{L+T}(\xi) = -\frac{131}{24} - \frac{9}{2}\ln\xi\,,\\
&\tilde{t}_0^{L+T}(\xi) = 0\,, && \tilde{t}_1^{L+T}(\xi) = 1\,, && \tilde{t}_2^{L+T}(\xi) = \frac{17}{2} + \frac{9}{2}\ln\xi\,,
\end{aligned}
\tag{C.6}
$$

while the m^4 terms have been only computed to $O(a)$ [25, 63, 64, 66, 67]:

$$
\begin{aligned}
&\tilde{h}_0^{L+T}(\xi) = 1 + \ln\xi\,, && \tilde{h}_1^{L+T}(\xi) = \frac{25}{4} - 2\zeta_3 + \frac{25}{3}\ln\xi + 4\ln^2\xi\,,\\
&\tilde{k}_0^{L+T}(\xi) = 0\,, && \tilde{k}_1^{L+T}(\xi) = 1 + \frac{4}{5}\ln\xi\,,\\
&\tilde{g}_0^{L+T}(\xi) = 1\,, && \tilde{g}_1^{L+T}(\xi) = \frac{94}{9} - \frac{4}{3}\zeta_3 + 8\ln\xi\,,\\
&\tilde{j}_0^{L+T}(\xi) = 0\,, && \tilde{j}_1^{L+T}(\xi) = 0\,,\\
&\tilde{u}_0^{L+T}(\xi) = 0\,, && \tilde{u}_1^{L+T}(\xi) = 0\,,\\
&\tilde{h}_0^{L}(\xi) = 1 + \ln\xi\,, && \tilde{h}_1^{L}(\xi) = \frac{41}{6} - 2\,\zeta_3 + \frac{28}{3}\ln\xi + 4\,\ln^2\xi\,,\\
&\tilde{k}_0^{L}(\xi) = 1 + \tfrac{2}{3}\,\ln\xi\,, && \tilde{k}_1^{L}(\xi) = 8 - \frac{4}{3}\zeta_3 + \frac{80}{9}\ln\xi + \frac{8}{3}\,\ln^2\xi\,.\\
&\tilde{j}_0^{L}(\xi) = 0\,, && \tilde{j}_1^{L}(\xi) = 0\,.
\end{aligned}
\tag{C.7}
$$

The scale dependence of all these coefficients is fixed by the homogeneous RG equations satisfied by the corresponding $D^J(s)$ functions:

$$
\begin{aligned}
&\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{p}_{n\geq 2}^{L+T}(\xi) = \sum_{k=1}^{n-1}(2k-n)\,\beta_k\,\tilde{p}_{n-k}^{L+T}(\xi)\,, && \frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{p}_1^{L+T}(\xi) = 0\,,\\
&\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{r}_{n\geq 3}^{L+T}(\xi) = \sum_{k=1}^{n-2}(k-n)\,\beta_k\,\tilde{r}_{n-k}^{L+T}(\xi) + \frac{2}{3}\sum_{k=1}^{n-1}k\,\gamma_k\,\tilde{p}_{n-k}^{L+T}(\xi)\,,\\
&\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{r}_2^{L+T}(\xi) = \frac{2}{3}\gamma_1\,\tilde{p}_1^{L+T}(\xi)\,,\\
&\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{q}_{n\geq 1}^{L+T}(\xi) = \sum_{k=1}^{n}(k-n)\,\beta_k\,\tilde{q}_{n-k}^{L+T}(\xi)\,, && \frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{q}_0^{L+T}(\xi) = 0\,,\\
&\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{t}_{n\geq 2}(\xi) = \sum_{k=1}^{n-1}(k-n)\,\beta_k\,\tilde{t}_{n-k}^{L+T}(\xi)\,, && \frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{t}_1^{L+T}(\xi) = 0\,,
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{h}^{L+T}_{n\geq 1}(\xi) &= \sum_{k=1}^{n}\left[4\gamma_k-(n-k)\,\beta_k\right]\tilde{h}^{L+T}_{n-k}(\xi)-\frac{1}{2}\sum_{k=0}^{n}\gamma^{ii}_{0(k)}\,\tilde{q}^{L+T}_{n-k}(\xi)\,,\\
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{h}^{L+T}_{0}(\xi) &= -\frac{1}{2}\,\gamma^{ii}_{0(0)}\,\tilde{q}^{L+T}_{0}(\xi)\,,\\
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{k}^{L+T}_{n\geq 2}(\xi) &= \sum_{k=1}^{n-1}\left[4\gamma_k-(n-k)\,\beta_k\right]\tilde{k}^{L+T}_{n-k}(\xi)-\frac{2}{5}\sum_{k=0}^{n-1}\gamma^{ii}_{0(k)}\,\tilde{t}^{L+T}_{n-k}(\xi)\,,\\
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{k}^{L+T}_{1}(\xi) &= -\frac{2}{5}\gamma^{ii}_{0(0)}\,\tilde{t}^{L+T}_{1}(\xi)\,,\\
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{g}^{L+T}_{n\geq 1}(\xi) &= \sum_{k=1}^{n}\left[4\gamma_k-(n-k)\,\beta_k\right]\tilde{g}^{L+T}_{n-k}(\xi)\,,\qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{g}^{L+T}_{0}(\xi)=0\\
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{j}^{L+T}_{n\geq 3}(\xi) &= \sum_{k=1}^{n-2}\left[4\gamma_k-(n-k)\,\beta_k\right]\tilde{j}^{L+T}_{n-k}(\xi)+\frac{1}{24}\sum_{k=1}^{n-1}k\,\gamma^{ii}_{0(k)}\,\tilde{p}^{L+T}_{n-k}(\xi)-\\
&\quad-\frac{1}{4}\sum_{k=0}^{n-2}\gamma^{ii}_{0(k)}\,\tilde{r}^{L+T}_{n-k}(\xi)\,,\qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{j}^{L+T}_{n\leq 1}(\xi)=0\,,\\
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{j}^{L+T}_{2}(\xi) &= \frac{1}{24}\gamma^{ii}_{0(1)}\,\tilde{p}^{L+T}_{1}(\xi)-\frac{1}{4}\gamma^{ii}_{0(0)}\,\tilde{r}^{L+T}_{2}(\xi)\,,\\
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{u}^{L+T}_{n\geq 3}(\xi) &= \sum_{k=1}^{n-2}\left[4\gamma_k-(n-k)\,\beta_k\right]\tilde{u}^{L+T}_{n-k}(\xi)+\frac{1}{24}\sum_{k=2}^{n-1}k\,\gamma^{i\neq j}_{0(k)}\,\tilde{p}^{L+T}_{n-k}(\xi)\\
&\qquad\qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{u}^{L+T}_{n\leq 2}(\xi)=0\,, \qquad\qquad (\mathrm{C.9})
\end{aligned}
$$

$$
\begin{aligned}
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{h}^{L}_{n\geq 1}(\xi) &= -\frac{1}{2}\gamma^{ii}_{0(n)}+\sum_{k=1}^{n}\left[4\gamma_k-(n-k)\,\beta_k\right]\tilde{h}^{L}_{n-k}(\xi)\,,\\
&\quad \xi\,\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{h}^{L}_{0}(\xi)=-\frac{1}{2}\,\gamma^{ii}_{0(0)}\,,\\
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{k}^{L}_{n\geq 1}(\xi) &= -\frac{1}{3}\,\gamma^{ii}_{0(n)}+\sum_{k=1}^{n}\left[4\gamma_k-(n-k)\,\beta_k\right]\tilde{k}^{L}_{n-k}(\xi)\,,\\
&\qquad\qquad \xi\,\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{k}^{L}_{0}(\xi)=-\frac{1}{3}\,\gamma^{ii}_{0(0)}\,,\\
\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{j}^{L}_{n\geq 3}(\xi) &= \sum_{k=1}^{n-2}\left[4\gamma_k-(n-k)\,\beta_k\right]\tilde{j}^{L}_{n-k}(\xi)\,,\qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\,\tilde{j}^{L}_{n\leq 2}(\xi)=0\,. \qquad (\mathrm{C.10})
\end{aligned}
$$

D. Scale evolution of the $D = 4$ operators

The factors $\gamma_{0(0)}^{ij}$ appearing in the RG equations (C.9) and (C.10) are the anomalous dimensions of the QCD vacuum energy,

$$4\,E_0 \equiv -\Theta_\mu^\mu + \sum_{k=u,d,s} m_k \overline{q}_k q_k\,, \tag{D.1}$$

where Θ_μ^μ is the trace of the QCD energy-momentum tensor, in the three light flavour effective theory, and

$$\mu\frac{\mathrm{d}E_0}{\mathrm{d}\mu} \equiv \frac{3}{16\pi^2}\sum_{i,j=u,d,s} m_i^2\, m_j^2\, \gamma_0^{ij}(a)\,; \qquad \gamma_0^{ij}(a) = \gamma_{0(0)}^{ij} + a\gamma_{0(1)}^{ij} + \cdots \tag{D.2}$$

The anomalous dimension matrix $\gamma_0(a)$ is symmetric, i.e. $\gamma_0^{ij}(a) = \gamma_0^{ji}(a)$. Moreover, since QCD is flavour blind, $\gamma_0^{ii}(a) = \gamma_0^{jj}(a)$ and all non-diagonal elements $\gamma_0^{i\neq j}(a)$ are equal. To two loops, $\gamma_0^{ij}(a)$ is proportional to the identity with [46]

$$\gamma_{0(0)}^{ii} = -2\,, \qquad \gamma_{0(1)}^{ii} = -\frac{8}{3}\,, \tag{D.3}$$

for $i = u, d, s$. The first non-diagonal terms appear at three-loops.

The anomalous dimension matrix $\gamma_0(a)$ governs the scale evolution of the $D = 4$ operators. After using the QCD equations of motion, there are three types of gauge-invariant operators of dimension four, namely, $G^2 \equiv G_{(a)}^{\mu\nu} G_{\mu\nu}^{(a)}$, $m\overline{q}q$, and m^4. In minimal subtraction-like schemes, these three operators mix under renormalization [46]:

$$\begin{aligned}
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\langle G^2\rangle &= -a\frac{\partial\beta(a)}{\partial a}\langle G^2\rangle + \frac{3}{4\pi^2}a\sum_{i,j=u,d,s} m_i^2 m_j^2 \frac{\partial\gamma_0^{ij}(a)}{\partial a} - 4a\frac{\partial\gamma(a)}{\partial a}\sum_{i=u,d,s}\langle m_i\overline{q}_i q_i\rangle\,,\\
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\langle m_i\overline{q}_j q_j\rangle &= -\frac{3}{4\pi^2} m_i\, m_j \sum_k m_k^2\, \gamma_0^{jk}(a)\,,\\
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\, m^4 &= -4\,\gamma(a)\, m^4\,.
\end{aligned} \tag{D.4}$$

One can introduce [46] the following scale-invariant condensates, which are combinations of the previous minimally subtracted operators:

$$\beta_1\langle aG^2\rangle_I \equiv \beta(a)\,\langle G_{(a)}^{\mu\nu} G_{\mu\nu}^{(a)}\rangle + 4\gamma(a)\sum_{i=u,d,s}\langle m_i\overline{q}_i q_i\rangle - \frac{3}{4\pi^2}\sum_{i,j=u,d,s} m_i^2\, m_j^2\, \gamma_0^{ij}(a)\,, \tag{D.5}$$

and

$$\begin{aligned}
\langle m_i\overline{q}_j q_j\rangle_I &\equiv \langle m_i\overline{q}_j q_j\rangle - \frac{3\,m_i\,m_j^3}{4\pi^2 a}\left\{\frac{\gamma_{0(0)}^{ii}}{\beta_1 + 4\gamma_1} + \left[\frac{\gamma_{0(1)}^{ii}}{4\gamma_1} - \frac{\gamma_{0(0)}^{ii}}{4\gamma_1}\,\frac{\beta_2 + 4\gamma_2}{\beta_1 + 4\gamma_1}\right] a\cdots\right\}\\
&= \langle m_i\overline{q}_j q_j\rangle + \frac{3\,m_i\,m_j^3}{7\pi^2 a}\left\{1 - \frac{53}{24}a + \cdots\right\}\,.
\end{aligned} \tag{D.6}$$

From these two invariants, one inmediately gets the scale evolution of the corresponding $\overline{\mathrm{MS}}$ operators. In particular, the SU(3)-breaking difference of quark condensates (4.14) satisfies:

$$\delta O_4(\mu^2) = \delta O_4(M_\tau^2) + \frac{3}{7\pi^2}\left(1-\epsilon_d^4\right) m_s^4(M_\tau^2) \times \\ \times\left\{\frac{1}{a_\tau}\left[1-\frac{53}{24}a_\tau+\cdots\right] - \frac{m_s^4(\mu^2)}{m_s^4(M_\tau^2)}\frac{1}{a(\mu^2)}\left[1-\frac{53}{24}a(\mu^2)+\cdots\right]\right\}. \quad \text{(D.7)}$$

E. $D = 4$ expansion coefficients for δR_τ^{kl}

The functions $Q_{kl}(a) = Q_{kl}^L(a) + Q_{kl}^{L+T}(a)$, $T_{kl}(a) = T_{kl}^L(a) + T_{kl}^{L+T}(a)$ and $S_{kl}(a) = S_{kl}^L(a) + S_{kl}^{L+T}(a)$ in eq. (4.15) are given by the following contour integrals

$$\begin{aligned}
Q_{kl}^L &= \frac{-1}{6\pi i}\oint_{|x|=1}\frac{dx}{x^2}\,\mathcal{F}_L^{kl}(x)\,\frac{\delta O_4(-\xi^2M_\tau^2x)}{\delta O_4(M_\tau^2)}\,,\\
Q_{kl}^{L+T} &= \frac{1}{4\pi i}\sum_n \tilde{q}_n^{L+T}(\xi)\oint_{|x|=1}\frac{dx}{x^3}\,\mathcal{F}_{L+T}^{kl}(x)\,\frac{\delta O_4(-\xi^2M_\tau^2x)}{\delta O_4(M_\tau^2)}\,a^n(-\xi^2M_\tau^2x)\,,\\
T_{kl}^L &= \frac{-1}{3\pi i}\sum_n\left[\tilde{h}_n^L(\xi)+\tilde{j}_n^L(\xi)\right]\oint_{|x|=1}\frac{dx}{x^2}\,\mathcal{F}_L^{kl}(x)\left(\frac{m(-\xi^2M_\tau^2x)}{m(M_\tau^2)}\right)^4 a^n(-\xi^2M_\tau^2x)\,,\\
T_{kl}^{L+T} &= \frac{1}{2\pi i}\sum_n \tilde{h}_n^{L+T}(\xi)\oint_{|x|=1}\frac{dx}{x^3}\,\mathcal{F}_{L+T}^{kl}(x)\left(\frac{m(-\xi^2M_\tau^2x)}{m(M_\tau^2)}\right)^4 a^n(-\xi^2M_\tau^2x)\,,\\
S_{kl}^L &= \frac{-1}{6\pi i}\sum_n\left[3\tilde{k}_n^L(\xi)-2\tilde{h}_n^L(\xi)-\tilde{j}_n^L(\xi)\right]\times\\
&\quad\times\oint_{|x|=1}\frac{dx}{x^2}\,\mathcal{F}_L^{kl}(x)\left(\frac{m(-\xi^2M_\tau^2x)}{m(M_\tau^2)}\right)^4 a^n(-\xi^2M_\tau^2x)\,,\\
S_{kl}^{L+T} &= \frac{1}{4\pi i}\sum_n \tilde{g}_n^{L+T}(\xi)\oint_{|x|=1}\frac{dx}{x^3}\,\mathcal{F}_{L+T}^{kl}(x)\left(\frac{m(-\xi^2M_\tau^2x)}{m(M_\tau^2)}\right)^4 a^n(-\xi^2M_\tau^2x)\,. \qquad \text{(E.1)}
\end{aligned}$$

To be consistent with our estimate of $\delta O_4(\mu^2)$, which is scale independent, we will also neglect the $\delta O_4(\mu^2)$ scale dependence in the functions $Q_{kl}^J(a)$. In the following tables we summarize the known values of the different $D = 4$ perturbative expansions for the (k,l) values used in the paper. We also give their numerical values obtained from the exact integration along the closed contour, using the corresponding RG equations.

(k,l)	$T_{kl}(a)$	$S_{kl}(a)$	$Q_{kl}(a)$
(0,0)	$1+\left(\frac{65}{6}-4\zeta_3-\frac{2}{3}\pi^2\right)a$	$1+\frac{19}{3}a$	$1+\frac{27}{8}a^2$
(1,0)	$\frac{5}{2}\left[1+\left(\frac{281}{30}-\frac{12}{5}\zeta_3-\frac{2}{5}\pi^2\right)a\right]$	$\frac{3}{2}\left[1+\left(\frac{244}{27}-\frac{4}{9}\zeta_3\right)a\right]$	$\frac{3}{2}\left[1-\frac{1}{3}a+\frac{13}{72}a^2\right]$
(2,0)	$\frac{25}{6}\left[1+\left(\frac{1381}{150}-\frac{48}{25}\zeta_3-\frac{8}{25}\pi^2\right)a\right]$	$2\left[1+\left(\frac{193}{18}-\frac{2}{3}\zeta_3\right)a\right]$	$2\left[1-\frac{1}{2}a-\frac{163}{96}a^2\right]$
(1,1)	$-\frac{5}{3}\left[1+\left(\frac{269}{30}-\frac{6}{5}\zeta_3-\frac{1}{5}\pi^2\right)a\right]$	$-\frac{1}{2}\left[1+\left(\frac{142}{9}-\frac{4}{3}\zeta_3\right)a\right]$	$-\frac{1}{2}\left[1-a-\frac{22}{3}a^2\right]$
(1,2)	$\frac{1}{8}\left[1+\frac{226}{15}a\right]$	$\frac{1}{2}a$	$-\frac{63}{160}a^2$

Table 6: Known terms of the relevant $D = 4$ perturbative expansions.

(k,l)	$T^L_{kl}(a)$	$S^L_{kl}(a)$	$Q^L_{kl}(a)$
(0,0)	$\frac{5}{2}\left[1+\left(\frac{28}{3}-\frac{8}{5}\zeta_3-\frac{4}{15}\pi^2\right)a\right]$	$1+\frac{37}{3}a$	1
(1,0)	$\frac{17}{6}\left[1+\left(\frac{496}{51}-\frac{24}{17}\zeta_3-\frac{4}{17}\pi^2\right)a\right]$	$1+\frac{41}{3}a$	1
(2,0)	$\frac{37}{12}\left[1+\left(\frac{1115}{111}-\frac{48}{37}\zeta_3-\frac{8}{37}\pi^2\right)a\right]$	$1+\frac{44}{3}a$	1
(1,1)	$-\frac{1}{4}\left[1+\frac{41}{3}a\right]$	$-a$	0
(1,2)	$-\frac{1}{20}\left[1+\frac{157}{15}a\right]$	$-\frac{1}{5}a$	0

Table 7: Known terms of the relevant $D=4$ perturbative expansions for $J=L$.

(k,l)	$T^{L+T}_{kl}(a)$	$S^{L+T}_{kl}(a)$	$Q^{L+T}_{kl}(a)$
(0,0)	$-\frac{3}{2}\left[1+\frac{25}{3}a\right]$	$-6a$	$\frac{27}{8}a^2$
(1,0)	$-\frac{1}{3}\left[1+\left(\frac{149}{12}+6\zeta_3+\pi^2\right)a\right]$	$\frac{1}{2}\left[1-\left(\frac{2}{9}+\frac{4}{3}\zeta_3\right)a\right]$	$\frac{1}{2}\left[1-a+\frac{13}{24}a^2\right]$
(2,0)	$\frac{13}{12}\left[1+\left(\frac{266}{39}-\frac{48}{13}\zeta_3-\frac{8}{13}\pi^2\right)a\right]$	$1+\left(\frac{61}{9}-\frac{4}{3}\zeta_3\right)a$	$1-a-\frac{163}{48}a^2$
(1,1)	$-\frac{17}{12}\left[1+\left(\frac{415}{51}-\frac{24}{17}\zeta_3-\frac{4}{17}\pi^2\right)a\right]$	$-\frac{1}{2}\left[1+\left(\frac{124}{9}-\frac{4}{3}\zeta_3\right)a\right]$	$-\frac{1}{2}\left[1-a-\frac{22}{3}a^2\right]$
(1,2)	$\frac{7}{40}\left[1+\frac{1444}{105}a\right]$	$\frac{7}{10}a$	$-\frac{63}{160}\,a^2$

Table 8: Known terms of the relevant $D=4$ perturbative expansions for $J=L+T$.

(k,l)	$T^L_{kl}(a_\tau)$	$S^L_{kl}(a_\tau)$	$Q^L_{kl}(a_\tau)$
(0,0)	$3.10\pm0.40\pm0.04$	$2.1\pm0.7\pm0.2$	$1.0\pm0.0\pm0.0$
(1,0)	$3.80\pm0.40\pm0.08$	$2.7\pm1.0\pm0.2$	$1.0\pm0.0\pm0.0$
(2,0)	$4.4\pm0.3\pm0.2$	$3.2\pm1.3\pm0.2$	$1.0\pm0.0\pm0.0$
(1,1)	$-0.66\pm0.07\pm0.06$	$-0.6\pm0.2\pm0.1$	$0.0\pm0.0\pm0.0$
(1,2)	$-0.03\pm0.10\pm0.07$	$-0.01\pm0.02\pm0.01$	$0.0\pm0.0\pm0.0$

Table 9: Numerical values of the relevant $D=4$ perturbative expansions for $\alpha_s(M_\tau^2)=0.35\pm0.02$ for $J=L$. The first error shows the estimated theoretical uncertainties taking $\alpha_s=0.35$; the second one shows the changes induced by the present uncertainty in the strong coupling.

(k,l)	$T^{L+T}_{kl}(a_\tau)$	$S^{L+T}_{kl}(a_\tau)$	$Q^{L+T}_{kl}(a_\tau)$
(0,0)	$-1.60\pm0.06\pm0.1$	$-1.1\pm0.5\pm0.1$	$0.08\pm0.03\pm0.01$
(1,0)	$-0.6\pm0.4\pm0.2$	$-0.6\pm0.4\pm0.1$	$0.52\pm0.03\pm0.01$
(2,0)	$0.80\pm0.60\pm0.05$	$0.25\pm0.05\pm0.05$	$0.93\pm0.02\pm0.003$
(1,1)	$-1.30\pm0.20\pm0.15$	$-0.86\pm0.18\pm0.02$	$-0.41\pm0.02\pm0.005$
(1,2)	$0.48\pm0.20\pm0.20$	$0.3\pm0.1\pm0.1$	$-0.02\pm0.01\pm0.003$

Table 10: Numerical values of the relevant $D=4$ perturbative expansions for $\alpha_s(M_\tau^2)=0.35\pm0.02$ for $J=L+T$. The first error shows the estimated theoretical uncertainties taking $\alpha_s=0.35$; the second one shows the changes induced by the present uncertainty in the strong coupling.

References

[1] Particle Data Group, *Eur. Phys. J.* **C 3** (1998) 1.

[2] Proceedings of the Fifth Workshop on Tau Lepton Physics — TAU 98 — (Santander, 14–17 September 1998), A. Pich and A. Ruiz eds., *Nucl. Phys.* **76** (1999).

[3] A. Pich, *Tau physics*, in *Heavy flavours II*, A.J. Buras and M. Lindner eds., Advanced Series on Directions in High Energy Physics – Vol. 15 World Scientific, Singapore 1998, p. 453 [`hep-ph/9704453`].

[4] ALEPH collaboration, *Study of τ decays involving kaons, spectral functions and determination of the strange quark mass*, `hep-ex/9903015`; *One-prong τ decays with kaons*, `hep-ex/9903014`.

[5] A. Höcker, *Nucl. Phys.* **55C** *(Proc. Suppl.)* (1997) 379 [`hep-ex/9703004`];
M. Davier, *Nucl. Phys.* **55C** *(Proc. Suppl.)* (1997) 395;
S. Chen, *Nucl. Phys.* **64** *(Proc. Suppl.)* (1998) 265;
S. Chen, M. Davier and A. Höcker, *Nucl. Phys.* **76** *(Proc. Suppl.)* (1999) 369.

[6] A. Pich and J. Prades, *J. High Energy Phys.* **06** (1998) 013 [`hep-ph/9804462`]; *Nucl. Phys.* **74** *(Proc. Suppl.)* (1999) 309 [`hep-ph/9811263`]; J. Prades, *Nucl. Phys.* **76** *(Proc. Suppl.)* (1999) 341.

[7] K.G. Chetyrkin, J.H. Kühn and A.A. Pivovarov, *Nucl. Phys.* **B 533** (1998) 473 [`hep-ph/9805335`].

[8] E. Braaten, S. Narison and A. Pich, *Nucl. Phys.* **B 373** (1992) 581.

[9] S. Narison and A. Pich, *Phys. Lett.* **B 211** (1988) 183.

[10] E. Braaten, *Phys. Rev. Lett.* **60** (1988) 1606; *Phys. Rev.* **D 39** (1989) 1458.

[11] E.C. Poggio, H.R. Quinn and S. Weinberg, *Phys. Rev.* **D 13** (1976) 1958.

[12] W. Marciano and A. Sirlin, *Phys. Rev. Lett.* **61** (1988) 1815; *Phys. Rev. Lett.* **56** (1986) 22.

[13] E. Braaten and C.S. Li, *Phys. Rev.* **D 42** (1990) 3888.

[14] F. Le Diberder and A. Pich, *Phys. Lett.* **B 286** (1992) 147.

[15] C.N. Lowett-Turner and C.J. Maxwell, *Nucl. Phys.* **B 452** (1995) 188 [`hep-ph/9505224`];
P. Ball, M. Beneke and V.M. Braun, *Nucl. Phys.* **B 452** (1995) 563 [`hep-ph/9502300`];
M. Neubert, *Nucl. Phys.* **B 463** (1996) 511 [`hep-ph/9509432`].

[16] F. Le Diberder and A. Pich, *Phys. Lett.* **B 289** (1992) 165.

[17] H. Leutwyler, in *Proc. of the NATO Advanced Study Institute on Masses of Fundamental Particles*, Cargèse, 1996, M. Lévy, J. Illiopoulos, R. Gastmans and J.-M. Gérard eds., NATO ASI Series B: Physics, Vol. 363 Plenum Press, New York, 1997, p.149 [hep-ph/9609467].

[18] J. Gasser and H. Leutwyler, *Nucl. Phys.* **B 250** (1985) 465; *Ann. Phys. (NY)* **158** (1984) 142.

[19] J. Bijnens, J. Prades and E. de Rafael, *Phys. Lett.* **B 348** (1995) 226 [hep-ph/9411285];
J. Prades, *Nucl. Phys.* **64** *(Proc. Suppl.)* (1998) 253 [hep-ph/9708395].

[20] S. Narison,*QCD spectral sum rules*, World Scientific Lecture Notes in Physics , Vol. 26 1989.

[21] H.G. Dosch, M. Jamin and S. Narison, *Phys. Lett.* **B 220** (1989) 251.

[22] S. Narison, *Phys. Lett.* **B 358** (1995) 113 [hep-ph/9504333].

[23] L.-E. Adam and K.G. Chetyrkin, *Phys. Lett.* **B 329** (1994) 129 [hep-ph/9404331].

[24] L.V. Lanin, V.P. Spiridonov and K.G. Chetyrkin, *Sov. J. Nucl. Phys.* **44** (1986) 892.

[25] M. Jamin and M. Münz, *Z. Physik* **C 60** (1993) 569 [hep-ph/9208201].

[26] M. Jamin and M. Münz, *Z. Physik* **C 66** (1995) 633 [hep-ph/9409335].

[27] D.J. Broadhurst and S.C. Generalis, *Phys. Lett.* **B 165** (1985) 175.

[28] D.J. Broadhurst and S.C. Generalis, *Phys. Lett.* **B 139** (1984) 85.

[29] W. Hubschmid and S. Mallik, *Nucl. Phys.* **B 207** (1982) 29.

[30] S.C. Generalis, *J. Phys.* **G 6** (1990) 367.

[31] ALEPH collaboration *Eur. Phys. J.* **C 4** (1998) 409.

[32] OPAL collaboration, *Eur. Phys. J.* **C 7** (1999) 571 [hep-ex/9808019].

[33] L. Lellouch, E. de Rafael and J. Taron, *Phys. Lett.* **B 414** (1997) 195 [hep-ph/9707523];
L. Lellouch, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 272 [hep-ph/9709405];
E. de Rafael, *Nucl. Phys.* **64** *(Proc. Suppl.)* (1998) 258 [hep-ph/9709430].

[34] F.J. Ynduráin, *Nucl. Phys.* **B 517** (1998) 324 [hep-ph/9708300].

[35] H.G. Dosch and S. Narison, *Phys. Lett.* **B 417** (1998) 173 [hep-ph/9709215].

[36] D. Becirevic, *Report on light quark masses.* Plenary talk at Lattice' 99, July 1999, Pisa, Italy;
R.D. Kenway, *Nucl. Phys.* **73** *(Proc. Suppl.)* (1999) 16 [hep-lat/9810054].

[37] C.R. Allton, *Nucl. Phys.* **64** *(Proc. Suppl.)* (1998) 242 [hep-ph/9710212].

[38] P. Colangelo, F. De Fazio, G. Nardulli and N. Paver, *Phys. Lett.* **B 408** (1997) 340 [hep-ph/9704249].

[39] M. Jamin, *Nucl. Phys.* **64** *(Proc. Suppl.)* (1998) 250 [hep-ph/9709484].

[40] C.A. Domínguez, L. Pirovano and K. Schilcher, *Phys. Lett.* **B 425** (1998) 193 [hep-ph/9712369].

[41] K. Maltman, *Phys. Lett.* **B 428** (1998) 179 [hep-ph/9804299]; *Phys. Rev.* **D 58** (1998) 093015 [hep-ph/9804298].

[42] K. Maltman, *The strange quark mass from finite energy sum rules*, hep-ph/9904370.

[43] S. Narison, *On the strange quark mass from* e^+e^- *data and test of the* SU(2) *isospin symmetry*, hep-ph/9905264.

[44] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, *Phys. Lett.* **B 400** (1997) 379 [hep-ph/9701390].

[45] K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, *Phys. Rev. Lett.* **79** (1997) 2184 [hep-ph/9706430].

[46] V.P. Spiridonov and K.G. Chetyrkin, *Sov. J. Nucl. Phys.* **47** (1988) 522.

[47] S.G. Gorishny, A.L. Kataev and S.A. Larin, *Phys. Lett.* **B 259** (1991) 144.

[48] J.A.M. Vermaseren, S.A. Larin and T. van Ritbergen, *Phys. Lett.* **B 405** (1997) 327 [hep-ph/9703284].

[49] K.G. Chetyrkin, *Phys. Lett.* **B 404** (1997) 161 [hep-ph/9703278].

[50] L.R. Surguladze, *Sov. J. Nucl. Phys.* **50** (1989) 372.

[51] S.G. Gosrishny, A.L. Kataev, S.A. Larin and L.R. Surguladze, *Mod. Phys. Lett.* **A 5** (1990) 2703.

[52] S.G. Gosrishny, A.L. Kataev, S.A. Larin and L.R. Surguladze, *Phys. Rev.* **D 43** (1991) 1633.

[53] S.G. Gorishny, A.L. Kataev and S.L. Larin, *Nuovo Cim.* **A92** (1986) 119.

[54] W. Bernreuther and W. Wetzel, *Z. Physik* **C 11** (1981) 113.

[55] K.G. Chetyrkin and J.H. Kühn, *Phys. Lett.* **B 406** (1997) 102 [hep-ph/9609202].

[56] K.G. Chetyrkin, *Phys. Lett.* **B 390** (1997) 309 [hep-ph/9608318].

[57] K.G. Chetyrkin and A. Kwiatkowski, *Z. Physik* **C 59** (1993) 525 [hep-ph/9805232]

[58] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys.* **B 147** (1979) 385; *Nucl. Phys.* **B 147** (1979) 448.

[59] G.T. Loladze, L.R. Surguladze, and F.V. Tkachov, *Phys. Lett.* **B 162** (1985) 363.

[60] C. Becchi, S. Narison, E. de Rafael, and F.J. Ynduráin, *Z. Physik* **C 8** (1981) 335.

[61] L.R. Surguladze and F.V. Tkachov, *Nucl. Phys.* **B 331** (1990) 35.

[62] E. Bagán, J.I. Latorre and P. Pascual, *Z. Physik* **C 32** (1986) 43.

[63] K.G. Chetyrkin, S.G. Gosrishny and V.P. Spiridonov, *Phys. Lett.* **B 160** (1985) 149.

[64] S.C. Generalis, *J. Phys.* **G 15** (1989) L225.

[65] P. Pascual and E. de Rafael, *Z. Physik* **C 12** (1982) 127.

[66] S.C. Generalis, *J. Phys.* **G 16** (1990) 785.

[67] D.J. Broadhurst and S.C. Generalis, *Improved two-loop mass corrections: the role of the quark condensate*, Open University preprint OUT-4102-22 1987 (unpublished).